Recall:

Level k gauged G/G WZW model has action:

$$k\,S_{G/G}(A,\lambda,g) = k\,S_G(A,g) - ik\,\Gamma(A,g) + \frac{i}{4\pi}\int_\Sigma \mathrm{Tr}(\lambda\wedge\lambda),$$

where

$$S_G(g,A) = -\frac{1}{8\pi}\int_\Sigma \mathrm{Tr}\left(g^{-1}d_A g \wedge * g^{-1}d_A g\right)$$

and

$$\Gamma(g,A) = \frac{1}{12\pi}\int_B \mathrm{Tr}\left[(g^{-1}dg)^3\right] - \frac{1}{4\pi}\int_\Sigma \mathrm{Tr}\left(A\,dg\,g^{-1} + A A^g\right),$$

$$A^g = g^{-1}Ag + g^{-1}dg\,, \quad \partial B = \Sigma$$

Action admits BRST-symmetry:

$$Q_g\,A = \lambda\,,$$

$$Q_g\,\lambda^{(1,0)} = (A^g)^{(1,0)} - A^{(1,0)}$$

$$Q_g\,\lambda^{(0,1)} = -(A^{g^{-1}})^{(0,1)} + A^{(0,1)}$$

$\rightarrow$ add chiral multiplet

$$\Phi = \varphi + \theta_\pm \psi^\pm + \theta^2 F$$

$\rightarrow$ perform top. twist by identifying

$$U(1)_{L^{\text{twist}}} \simeq \operatorname{diag}\left(U(1)_L \times U(1)_V\right)$$

$\rightarrow$ φ becomes a section of $\Omega^0(\Sigma, K^{R/2})$,

$\psi^{\pm}$ a section of $\Omega^0(\Sigma, K^{(R-1\pm 1)/2})$,

and F an element of $H^0(\Sigma, K^{R/2-1})$

Thus we get

$$(\varphi, \psi = \psi^+) \in \Gamma\left[\Omega^0(\Sigma, K^{R/2})\right]$$

$$(\chi = \psi^-, \eta = F) \in \Gamma\left[\Omega^0(\Sigma, K^{R/2-1})\right]$$

along with complex conjugates $(\varphi^\dagger, \psi^\dagger)$ and $(\chi^\dagger, \eta^\dagger)$ from $\Phi^\dagger$.

For $R=2$, (χ, η) are scalars while (φ, ψ) are $(1,0)$-forms

$\rightarrow$ parametrize deformations of M5-branes wrapped on $\Sigma \subset T^*\Sigma$.

For $R=0$, (φ, ψ) are scalars, while (χ, η) are $(0,1)$-forms.

$\rightarrow$ corresponds to the geometry $\Sigma \times \mathbb{C}$

This motivates the "equivariant G/G model" with general R:

- fields: $(A, \lambda, \varphi, \psi, \eta, \chi, g)$
  where A, φ, η, g are bosons and the rest are fermions
- action of BRST charge:

$$Q_{(q,t)} A = \lambda, \quad Q_{(q,t)} \lambda^{(1,0)} = (A^g)^{(1,0)} - A^{(1,0)},$$

$$Q_{(q,t)} \lambda^{(0,1)} = -(A^{g^{-1}})^{(0,1)} + A^{(0,1)},$$

$$Q_{(q,t)} \varphi = \psi, \quad Q_{(q,t)} \psi = t(\varphi^g) - \varphi,$$

$$Q_{(q,t)} \psi^\dagger = -t(\varphi^\dagger)^{g^{-1}} + \varphi^\dagger,$$

$$Q_{(q,t)} \chi = \eta, \quad Q_{(q,t)} \eta = t\chi^g - \chi,$$

$$Q_{(q,t)} \eta^\dagger = -t(\chi^\dagger)^{g^{-1}} + \chi^\dagger, \quad Q_{(q,t)} g = 0$$

where

$$A^g = g^{-1} A g + g^{-1} dg,$$

$$\varphi^g = g^{-1} \varphi g,$$

$$\chi^g = g^{-1} \chi g$$

The square of the BRST charge
$Q^2_{(q,t)} = \mathcal{L}_{(q,t)}$ defines a bosonic trf.
on the space of fields
action of gauged equivariant WZW model:

$$S_{R\text{-}EGWZW} = S_{GWZW} + Q_{(q,t)}(S') \quad (*)$$

where S' has to satisfy

$$\mathcal{L}_{(q,t)} S' = 0$$

Concretely, the second term of $(*)$
takes the form

$$S_{matter}(q, A, \varphi, \psi, \eta, \chi) = Q_{(q,t)} S'$$

$$= Q_{(q,t)} \left[\frac{1}{4\pi} \int_\Sigma \mathrm{Tr}\left(\varphi\psi^\dagger - \psi\varphi^\dagger + \chi\eta^\dagger - \eta\chi^\dagger \right) \right]$$

$$= \frac{1}{2\pi} \int_\Sigma \left\{ (\varphi - t\varphi^q, \varphi) + (\psi, \psi) + (\chi - t\chi^q, \chi) + (\eta, \eta) \right\}$$

Now use supersymmetric localization to compute path integral with action(*)!

$$Z = \int \mathcal{D}\{\text{fields}\}\ e^{-(S_{GWZW} + \frac{1}{\lambda} Q_{(q,t)} S')}$$

as $Q_{(q,t)} S'$ is exact in BRST-coh, we can sent $\lambda \to 0$

$\to$ path integral localizes on configurations corresponding to saddle points of $Q_{(q,t)} S'$ and one-loop fluctuations around it!

Concretely, one chooses a gauge

$$g = \exp\left(2\pi i \sum_{a=1}^{N} \sigma_a H^a\right) \quad \text{"abelian gauge"}$$

Cartan generators

$\to$ fields (A, λ, g) are replaced by abelian fields $(A_a, \lambda_a, \sigma_a)$

giving rise to non-trivial principal $U(1)^N$-bundle characterized by the flux $(n_1, \dots, n_N)$:

$$n_a = \frac{1}{2\pi} \int_\Sigma F_a$$

→ need to sum over all flux sectors in path integral

Let us define

$$J_a(\sigma) = k\sigma_a - \frac{i}{2\pi} \sum_{b \neq a} \log\left(\frac{e^{2\pi i \sigma_a} - t e^{2\pi i \sigma_b}}{t e^{2\pi i \sigma_a} - e^{2\pi i \sigma_b}}\right)$$

and denote by {Bethe} the set of solutions to the equations:

$$e^{2\pi i k \sigma_a} \prod_{b \neq a} \left(\frac{e^{2\pi i \sigma_a} - t e^{2\pi i \sigma_b}}{t e^{2\pi i \sigma_a} - e^{2\pi i \sigma_b}}\right) = 1 \quad \forall a = 1, 2, \dots, N$$

Carrying out the path integral via localization then gives the partition function

$$Z^R(\Sigma; U(N), k, t)$$

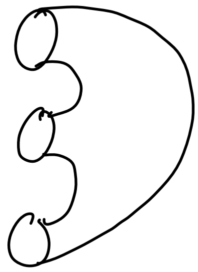
$$= \sum_{\{\sigma\} \in \{\text{Bethe}\}} \left[(1-t)^{N(1-R)} \det \left| \frac{\partial J_a}{\partial \sigma_b} \right| \prod_{a \neq b} \frac{(e^{2\pi i \sigma_a} - t e^{2\pi i \sigma_b})^{1-R}}{e^{2\pi i \sigma_a} - e^{2\pi i \sigma_b}} \right]^{h-1}$$

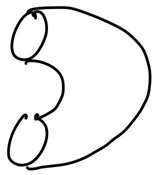
where h is the genus of the Riemann surface Σ

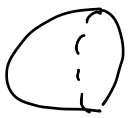
This is the partition function of a 2d TQFT!

Any 2d TQFT can be formulated in a set of Atiyah-Segal axioms:

- assign a Hilbert space V to a circle S^1
- assign $\mathrm{Hom}(V^{\otimes n}, \mathbb{C})$ to a punctured Riemann surface $\Sigma_{h,n}$
  - $\to$ for $n=0$ assign an element in $\mathrm{Hom}(\mathbb{C}, \mathbb{C})$ "partition function"
- any Riemann surface $\Sigma_{h,n}$ (n is number of punctures), can be decomposed into 3 basic ingrediends:

 $\rightsquigarrow f^{\mu\nu\rho}$

 $\rightsquigarrow \eta^{\mu\nu}$

 $\rightsquigarrow e_\phi$

where indices μ, ν, ρ correspond to basis vectors $|\mu\rangle$ of V

Topological invariance requires $f^{\mu\nu\rho}$ to be symmetric and satisfy:

$$f^{\mu_1 \nu_1 \rho_1} \eta_{\rho_1 \rho_2} f^{\mu_2 \nu_2 \rho_2} = f^{\mu_1 \nu_2 \rho_1} \eta_{\rho_1 \rho_2} f^{\mu_2 \nu_1 \rho_2}$$

In our specific case of EGWZW, we can determine the dimension of V by:

$$\dim V = Z_{EGWZW}[T^2; SU(2)] = \sum_{\{\text{Bethe}\}} 1$$

The Bethe ansatz equations for $SU(2)$ can be obtained by combining the two equations for $U(2)$,

$$e^{2\pi i k\sigma_1}\left(\frac{e^{2\pi i\sigma_1}-te^{2\pi i\sigma_2}}{te^{2\pi i\sigma_1}-e^{2\pi i\sigma_2}}\right)=1,$$

$$e^{2\pi i k\sigma_2}\left(\frac{e^{2\pi i\sigma_2}-te^{2\pi i\sigma_1}}{te^{2\pi i\sigma_2}-e^{2\pi i\sigma_1}}\right)=1,$$

into a single equation satisfied by

$$\sigma=\frac{1}{2}(\sigma_1-\sigma_2)\in\left[0,\frac{1}{2}\right]$$

$$\Rightarrow\ e^{4\pi i k\sigma}\left(\frac{e^{2\pi i\sigma}-te^{-2\pi i\sigma}}{te^{2\pi i\sigma}-e^{-2\pi i\sigma}}\right)^2=1$$

One can verify that the number of solutions is always $k+1$.